LETTER

An Approximation Algorithm for Two-Dimensional Warping*

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SUMMARY A new efficient two-dimensional warping algorithm is presented, in which sub-optimal warping is attained by iterating DP-based local optimization of warp on partially overlapping subplane sequence. From an experimental comparison with a conventional approximation algorithm based on beam search DP, relative superiority of the proposed algorithm is established.

key words: *two-dimensional warping, dynamic programming, approximation algorithm*

1. Introduction

Two-dimensional warping (2DW), defined as pixel-to-pixel mapping between given two images, is one of fundamental techniques for pattern recognition and image analysis problems. The authors have proposed a dynamic programming (DP) based algorithm for the monotonic and continuous 2DW [1]–[3]. The algorithm searches for *optimal* 2DW subject to two-dimensional monotonicity and continuity constraints for preserving topological structures in images. The disadvantage of the algorithm is its prohibitive computational complexity. Thus, for practical use, one has to resort to an approximation algorithm such as beam search DP [2], [3] at the cost of the optimality.

In this paper, a new approximation algorithm for the monotonic and continuous 2DW is presented. This algorithm is based on local and iterative use of the DP algorithm for the optimal 2DW. An experimental result shows the superiority of the present algorithm in efficiency over a conventional approximation algorithm based on beam search DP.

2. Monotonic and Continuous 2DW Problem and DP Algorithm [2], [3]

In this section, the monotonic and continuous 2DW problem is first formalized and then the DP algorithm for optimal 2DW is reviewed as preparations for discussing a new approximation algorithm in Sect. 3.

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2.1 Problem Formulation

Consider two images $\boldsymbol{A} = \{\boldsymbol{a}(i,j) \mid i = 1,\ldots,N_i,\ j = 1,\ldots,N_j\}$ and $\boldsymbol{B} = \{\boldsymbol{b}(x,y) \mid x = 1,\ldots,M_x,\ y = 1,\ldots,M_y\}$ where pixel values $\boldsymbol{a}(i,j)$ and $\boldsymbol{b}(x,y)$ may be vectors. The optimal monotonic and continuous 2DW between $\boldsymbol{A}$ and $\boldsymbol{B}$ is defined by the warping function $x = x(i,j)$, $y = y(i,j)$ which minimizes the following criterion function

$$\sum_{i=1}^{N_i}\sum_{j=1}^{N_j} \delta(\boldsymbol{a}(i,j),\ \boldsymbol{b}(x(i,j),y(i,j))) \tag{1}$$

where $\delta(\cdot,\cdot)$ is a distance function between two pixel values. The warping function is constrained by the two-dimensional monotonicity and continuity conditions defined as

$$0 \le x(i,j) - x(i-1,j) \le 2 \tag{2}$$
$$0 \le y(i,j) - y(i,j-1) \le 2 \tag{3}$$
$$|x(i,j) - x(i,j-1)| \le 1 \tag{4}$$
$$|y(i,j) - y(i-1,j)| \le 1 \tag{5}$$

and the boundary conditions defined as

$$x(1,j) = y(i,1) = 1 \tag{6}$$
$$x(N_i,j) = M_x \tag{7}$$
$$y(i,N_j) = M_y. \tag{8}$$

These constraints guarantee the 2DW to approximately preserve the topological structure in images. Figure 1 shows examples of the monotonic and continuous 2DW.

Let $D(\boldsymbol{A},\boldsymbol{B})$ denote the minimum value of (1). The quantity $D(\boldsymbol{A},\boldsymbol{B})$ is a distance between $\boldsymbol{A}$ and optimally deformed $\boldsymbol{B}$.

2.2 DP Algorithm for Optimal 2DW

The minimization problem of (1) subject to the constraints (2)–(8) can be solved by the DP. In the DP algorithm, the 2DW problem is treated as the optimal state transition sequence search problem on the following multistage decision process, specified by stage, decision, state transition, and transition cost.

Stage and decision: Consider a process where mapping of pixel (i,j), i.e., $(x(i,j),y(i,j))$ is decided one-by-one along the vertical raster scanning path. Since

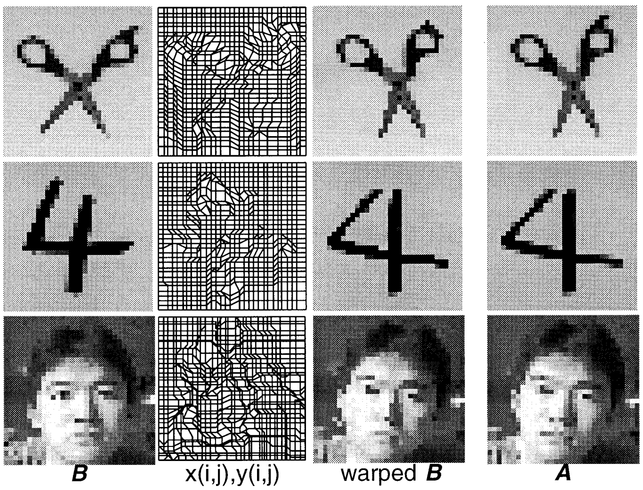

Fig. 1 Examples of the monotonic and continuous 2DW. ($N_i = N_j = M_x = M_y = 32$)

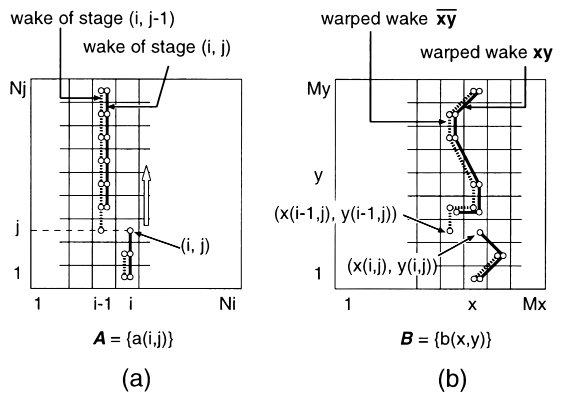

Fig. 2 Wakes and warped wakes.

decisions are mutually constrained by (2)–(5), the decision at stage (i, j) is constrained by the directly preceding decision as well as by the N_j stage preceding decision. For dealing with this N_jth-order Markovian nature, it is convenient to consider that each stage (i, j) is accompanied by the *wake* $[(i-1, j+1), \ldots, (i-1, N_j), (i, 1), \ldots, (i, j)]$ (Fig. 2 (a)).

State: The mapping of the wake of stage (i, j), called *warped wake*, is denoted as $\boldsymbol{xy}(i, j)$ (often abbreviated as $\boldsymbol{xy}$). The constraints (3), (4), and boundary conditions (6)–(8) hold in the individual warped wakes. Figure 2 (b) shows an example of warped wake. Each warped wake $\boldsymbol{xy}(i, j)$ corresponds to a state belonging to stage (i, j). Thus, the notation $\boldsymbol{xy}(i, j)$ is also used to represent the state. Let $\boldsymbol{XY}(i, j)$ denote the set of warped wakes, or states, at stage (i, j). Since we assume that the mapping of the first N_j pixels $(1, 1), \ldots, (1, N_j)$ is decided at stage $(1, N_j)$, $\boldsymbol{XY}(1, N)$ is the set of initial states. The final stage is (N_i, N_j) and $\boldsymbol{XY}(N_i, N_j)$ is the set of final states.

State transition: Let $\overline{\boldsymbol{xy}}$ be a state belonging to stage $(i, j-1)$. The transition from state $\overline{\boldsymbol{xy}}$ to $\boldsymbol{xy}$ is allowed (i.e., $\overline{\boldsymbol{xy}}$ and $\boldsymbol{xy}$ are consecutive) if and only if $\overline{\boldsymbol{xy}}$ and $\boldsymbol{xy}$ represent the same mapping except for $(x(i-1, j), y(i-1, j))$ of $\overline{\boldsymbol{xy}}$ and $(x(i, j), y(i, j))$ of $\boldsymbol{xy}$ and these two endpoints mutually satisfy the monotonicity and continuity constraints (2) and (5). Figure 2 (b) shows a warped wake $\boldsymbol{xy}$ and its preceding warped wake $\overline{\boldsymbol{xy}}$. Let $\overline{\boldsymbol{XY}}(\boldsymbol{xy})$ denote the set of the preceding states $\overline{\boldsymbol{xy}}$ of $\boldsymbol{xy}$. It should be noted that $|\overline{\boldsymbol{XY}}(\boldsymbol{xy})| \leq 9$.

Transition cost: Since the mapping of pixel (i, j), i.e., $(x(i, j), y(i, j))$ is decided at stage (i, j), the following cost is additively imposed to the transitions to a state $\boldsymbol{xy} \in \boldsymbol{XY}(i, j)$

$$d(i, j, \boldsymbol{xy}) = \delta(\boldsymbol{a}(i, j),\ \boldsymbol{b}(x(i, j), y(i, j))). \tag{9}$$

DP algorithm: Now the 2DW problem can be solved by searching all possible state transition sequences in the above decision process for one with minimum cumulative cost. This search can be performed by calculating DP-equation defined as

$$\begin{aligned} & g(i, j, \boldsymbol{xy}) \\ & \quad = d(i, j, \boldsymbol{xy}) \\ & \qquad + \min_{\overline{\boldsymbol{xy}} \in \overline{\boldsymbol{XY}}(\boldsymbol{xy})} \begin{cases} g(i-1, N_j, \overline{\boldsymbol{xy}}) & j = 1 \\ g(i, j-1, \overline{\boldsymbol{xy}}) & j \neq 1 \end{cases} \end{aligned} \tag{10}$$

at each state from stage $(2, 1)$ to (N_i, N_j), where

$$g(1, N_j, \boldsymbol{xy}) = \sum_{j=1}^{N_j} \delta(\boldsymbol{a}(1, j), \boldsymbol{b}(x(1, j), y(1, j))).$$

Then the image distance $D(\boldsymbol{A}, \boldsymbol{B})$ is obtaind as

$$D(\boldsymbol{A}, \boldsymbol{B}) = \min_{\boldsymbol{xy} \in \boldsymbol{XY}(N_i, N_j)} g(N_i, N_j, \boldsymbol{xy}). \tag{11}$$

The optimal warping function $x(i, j)$, $y(i, j)$ can be obtained through backtracking operations.

Computational complexity: The time complexity of the above DP algorithm is the product of the numbers of stages, states belonging to each stage, and possible transitions from each state, i.e., $O(N_i N_j) \times O(M_x 9^{N_j})$ $\times 9 = O(N_i N_j M_x 9^{N_j})$. Thus, if N_j is large, one has to resort to an approximation algorithm with less complexity.

3. Approximation Algorithm

In this section, a new approximation algorithm for the 2DW problem is presented. The present algorithm is developed by exploiting the fact that the complexity of the above DP algorithm mainly depends on N_j rather than N_i, M_x and M_y. Figure 3 illustrates this algorithm.

The present algorithm first searches for the optimal mapping of the $N_i \times K$ *block* which contains first K rows of the image $\boldsymbol{A}$, using the above DP algorithm. For this optimization, $O(N_i K M_x 9^K)$ computation is required. Thus, if K is small, this optimization

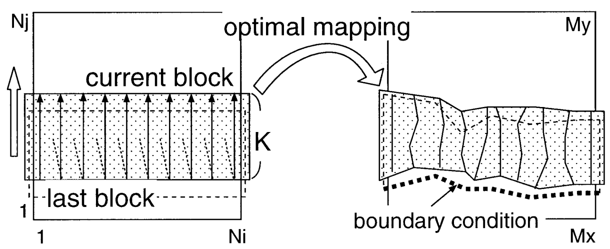

Fig. 3 The principle of the present approximation algorithm.

will be tractable, while the obtained 2DW is optimal within the first block. The obtained mapping of the first row, i.e., $((x(1,1), y(1,1)) \ldots, (x(N_i,1), y(N_i,1)))$, is kept and the rest is discarded. Second, the algorithm searches for the optimal mapping of the second block (rows 2 through $1+K$) using the DP algorithm. In this optimization, the mapping of the first row is used as boundary conditions. Repeating the similar operation until the $(N_j - K + 1)$th block is processed, a suboptimal 2DW between $\boldsymbol{A}$ and $\boldsymbol{B}$ is obtained. The computational complexity of the algorithm is $O((N_j - K + 1)N_i K M_x 9^K) = O(N_i N_j M_x K 9^K)$.

A similar approximation technique has been employed in Markov random field (MRF) based image segmentation [4]. It also should be noted that when $K = 1$, the present approximation algorithm is essentially equivalent to Sugimura et al. [5].

4. Experimental Result

An experiment was performed to compare the efficiency of the present algorithm with that of the beam search based approximation algorithm previously reported by the authors [2], [3]. In the previous algorithm, only first R states with the smallest cumulative costs g are taken into account as promising search paths at each stage.

The efficiency was evaluated by computation time and accuracy. The computation time of the present and previous algorithms was controlled by K and beam size R, respectively. The accuracy was measured by $D(\boldsymbol{A}, \boldsymbol{B})$. A smaller value of $D(\boldsymbol{A}, \boldsymbol{B})$ means a higher accuracy.

The experiment was performed on one hundred pairs of 256-level gray scale pseudorandom noise images and one hundred pairs of 256-level RGB images of human faces. The size of these images is 32, i.e., $N_i = N_j = M_x = M_y = 32$. L_1-norm was used as the distance function δ.

Figure 4 shows averaged $D(\boldsymbol{A}, \boldsymbol{B})$ as a function of averaged computation time. Clearly we see

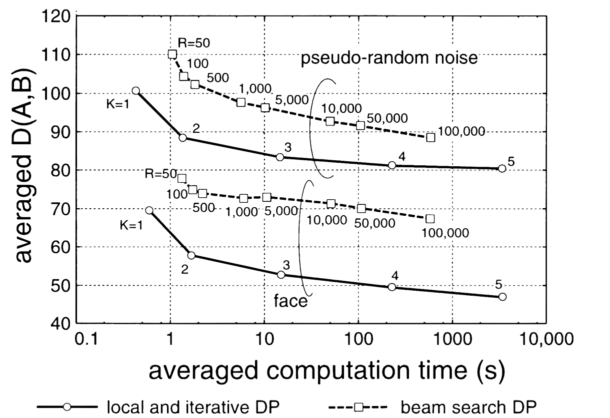

Fig. 4 Efficiency comparison between the present ("local and iterative DP") and previous ("beam search DP") approximation algorithms.

that the present algorithm attains a higher accuracy than the previous algorithm at the same computation time. In other words, the present algorithm requires far less computation time than the previous algorithm for the 2DW with the same accuracy.

5. Conclusion

For the monotonic and continuous 2DW, we presented a new approximation algorithm where sub-optimal warping is attained by iterating DP-based local optimization of warp on partially overlapping subplane sequence. An experimental result shows the superiority of the present algorithm in computational efficiency over a conventional approximation algorithm based on beam search DP.

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