# Piecewise Linear Two-Dimensional Warping

Seiichi Uchida and Hiroaki Sakoe
Graduate School of Information Science and Electrical Engineering, Kyushu Univ.
6-10-1 Hakozaki, Higashi-ku, Fukuoka-shi, 812-8581, Japan
{uchida, sakoe}@is.kyushu-u.ac.jp

## Abstract

*A new efficient dynamic programming (DP) algorithm for 2D elastic matching is proposed. The present DP algorithm requires by far less complexity than previous DP-based elastic matching algorithms. This complexity reduction results from piecewise linearization of a 2D-2D mapping which specifies an elastic matching between two given images. Since this linearization can be guided by a priori knowledge related to image patterns to be matched, the present DP algorithm often provides sufficient matching as is shown by experimental results.*

## 1. Introduction

Two-dimensional (2D) elastic matching, or deformable template is one of the most fundamental techniques for pattern recognition and image analysis [1, 2, 3, 4]. In 2D elastic matching, one image is linearly or nonlinearly warped and then fitted to another image by a 2D-2D mapping called 2D warping (2DW). Generally, 2DW is determined by solving a pixel-to-pixel correspondence optimization problem and therefore the characteristics of the 2DW depend on employed optimization strategy as well as specification for the mapping function.

*Dynamic programming* (DP) is one of the optimization strategies employed in 2DW problems. DP possesses several advantageous characteristics for the 2DW, such as 1) global optimality of its solution, 2) wide varieties of applicable constraints and criterion functions, 3) computational stability, etc. Levin and Pieraccini [5] has proposed a DP algorithm which provides *monotonic* 2DW. Unlike other DP-based 2DW's developed by simple combinations of orthogonal one-dimensional warpings, such as DP-based stereo [6] and PHMM [7], the monotonic 2DW can adjust the essentially 2D deformations. From practical viewpoints, however, there remained two serious problems, i.e., excessive deformation and prohibitive computations. In order to overcome these problems, Uchida and Sakoe [8, 9] has proposed another DP-based 2DW where *continuity* constraints as well as the monotonicity constraints are newly incorporated. Unfortunately, its computational complexity still remained in exponential order of image size and therefore one has to resort to approximation algorithms such as beam search DP [9] at the cost of the optimality.

In this paper, piecewise linear 2DW and its DP algorithm (PL2DW) are investigated. The present DP algorithm requires far less computations than the conventional monotonic and continuous 2DW algorithm. This complexity reduction results from piecewise linearization of the mapping. In the present PL2DW, the mapping of each column of one image onto another image is given by linear interpolation of the mapping of several points, called *pivots*, prepared on the column. While this linearization somewhat limits the fitting ability of the 2DW, the PL2DW provides some sufficient matching for wide class of image patterns. This is because the linearization can be guided by *a priori* knowledge related to image patterns to be matched. The knowledge is incorporated into the number and position of the pivots. Experimental results show that the PL2DW yields good matchings, if the appropriate pivots are prepared.

## 2. Piecewise Linear Two-Dimensional Warping

### 2.1. Pivot and piecewise linearization of mapping

Figure 1(a) illustrates the present PL2DW. Consider two $N \times N$ images $\boldsymbol{A} = \{a(i,j)\}$ and $\boldsymbol{B} = \{b(x,y)\}$. Let $x = x(i,j)$, $y = y(i,j)$ denote a 2D-2D mapping to be optimized. In the PL2DW, the mapping of $N$ points of each column of $\boldsymbol{A}$ onto $\boldsymbol{B}$ is given by linear interpolation of the mapping of $K$ *pivots* prepared on the column (Fig.2). In other words, the $i$th column of $\boldsymbol{A}$ is bended at $K(\leq N)$ points, and each piece between these points is linearly elongated or shortened, and then mapped to $\boldsymbol{B}$. Thus, at each column, the PL2DW is controlled by the mapping of $K$ points, while the conventional monotonic and continuous 2DW [8, 9] (Fig. 1(b)) and the monotonic 2DW [5] are controlled by the mapping of all $N$ points. As is discussed in

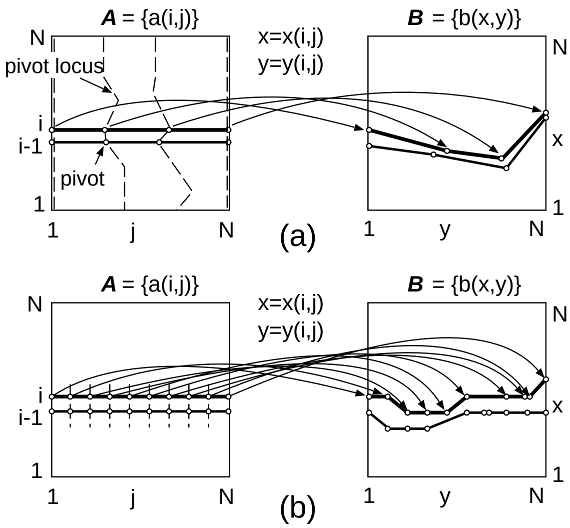


**Figure 1. Piecewise linear 2DW (PL2DW, $K = 4$) (a) and the conventional monotonic and continuous 2DW [8, 9] (b).**

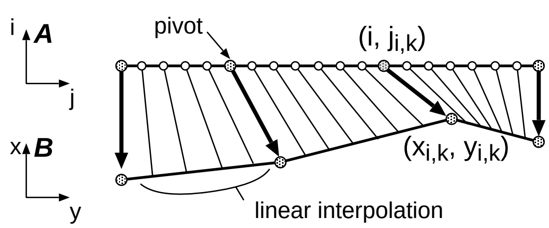


**Figure 2. Mapping of the $i$th column of image $\boldsymbol{A}$.**

**2.5**, this results in enormous difference in their computational complexity.

Let $j_{i,k}$ denote the $j$ coordinate of the $k$th pivot prepared on the $i$th column of $\boldsymbol{A}$ and assume

$$1 = j_{i,1} < j_{i,k-1} < j_{i,k} < j_{i,K} = N \tag{1}$$

and

$$|j_{i,k} - j_{i-1,k}| \leq 1. \tag{2}$$

The mapping of point $(i, j)$, $j \in [j_{i,k-1}, j_{i,k}]$ is given by the linear interpolation of pivot mapping $(x_{i,k}, y_{i,k}) = (x(i, j_{i,k}), y(i, j_{i,k}))$, i.e.,

$$\left.\begin{array}{rcl} x(i,j) & = & j' \cdot x_{i,k} + (1-j') \cdot x_{i,k-1} \\ y(i,j) & = & j' \cdot y_{i,k} + (1-j') \cdot y_{i,k-1} \end{array}\right\} \tag{3}$$

where $j' = (j - j_{i,k-1})/(j_{i,k} - j_{i,k-1})$ (Fig.2).

The pivots should be artificially prepared on $\boldsymbol{A}$ before the optimization of the PL2DW (i.e., the optimization of pivot mapping). In order to minimize the distortion caused by the piecewise linearization, it is desirable to arrange the pivots according to *a priori* knowledge related to the bending characteristics of image patterns to be matched. In other words, the PL2DW has the ability to obtain accurate matchings by incorporating the *a priori* knowledge into the pivot arrangement.

## 2.2. Constraints on mapping

Meaningful deformations generally preserve the topological structures of the image patterns. Therefore, the following monotonicity and continuity constraints are imposed on the pivot mapping

$$\left.\begin{array}{l} 0 \leq x_{i,k} - x_{i-1,k} \leq 2 \\ |y_{i,k} - y_{i-1,k}| \leq 1 \\ 0 \leq y_{i,k} - y_{i,k-1} \leq 2(j_{i,k} - j_{i,k-1}) \end{array}\right\}. \tag{4}$$

When these constraints hold, the pivots which form $K$ continuous loci on $\boldsymbol{A}$ by (2), are mapped on $\boldsymbol{B}$ resulting $K$ (approximately) continuous loci. It is clear that the monotonicity and continuity relation between non-pivot points is also preserved under these constraints.

Unless otherwise mentioned, the following boundary conditions are imposed on the mapping :

$$\left.\begin{array}{l} x_{1,k} = y_{i,1} = 1 \\ x_{N,k} = y_{i,K} = N \end{array}\right\}. \tag{5}$$

Additionally, if it is known that the deformation of the image patterns is relatively small, the warping accuracy can be improved by warp range limitation,

$$\left.\begin{array}{l} |x_{i,k} - i| \leq w \\ |y_{i,k} - j_{i,k}| \leq w \end{array}\right\} \tag{6}$$

where $w$ is a nonnegative constant specifying warp range.

## 2.3. Criterion function

Assume that the mapping of the $i$th column of image $\boldsymbol{A}$ is evaluated by

$$d((x_{i,1}, y_{i,1}), \ldots, (x_{i,k}, y_{i,k}), \ldots, (x_{i,K}, y_{i,K}) \mid i) = \sum_{j=j_{i,1}}^{j_{i,K}} |a(i,j) - b(x(i,j), y(i,j))|. \tag{7}$$

Then the optimal PL2DW is defined as the pivot mapping $\{(x_{i,k}, y_{i,k})\}$ which minimizes the criterion function

$$\sum_{i=1}^{N} d((x_{i,1}, y_{i,1}), \ldots, (x_{i,k}, y_{i,k}), \ldots, (x_{i,K}, y_{i,K}) \mid i)$$

subject to the constraints (4)–(6). Let $D(\boldsymbol{A}, \boldsymbol{B})$ denote the minimum value of the criterion function. The quantity

/* *Initialization* */

1 **for all** $[y_{1,2}, \ldots, y_{1,K-1}]$ where $|y_{1,k} - j_{1,k}| \le w$ **do**

2 $\quad g((1, y_{1,1}), \ldots, (1, y_{1,k}), \ldots, (1, y_{1,K}) \mid 1) := d((1, y_{1,1}), \ldots, (1, y_{1,k}), \ldots, (1, y_{1,K}) \mid 1)$

/* *Recursion* */

3 **for** $i := 2$ **to** $N$ **do**

4 $\quad$ **for all** $[(x_{i,1}, y_{i,1}), \ldots, (x_{i,k}, y_{i,k}), \ldots, (x_{i,K}, y_{i,K})]$ where $|x_{i,k} - i| \le w$ and $|y_{i,k} - j_{i,k}| \le w$ **do**

5 $$g((x_{i,1}, y_{i,1}), \ldots, (x_{i,k}, y_{i,k}), \ldots, (x_{i,K}, y_{i,K}) \mid i) := d((x_{i,1}, y_{i,1}), \ldots, (x_{i,k}, y_{i,k}), \ldots, (x_{i,K}, y_{i,K}) \mid i) + \min_{\substack{[(p_1,q_1),\ldots,(p_K,q_K)] \\ (p_k,q_k) \in \Delta_k}} g\big( (x_{i,1} - p_1, y_{i,1} - q_1), \ldots, (x_{i,k} - p_k, y_{i,k} - q_k), \ldots, (x_{i,K} - p_K, y_{i,K} - q_K) \mid i - 1 \big)$$

/* *Termination* */

6 $$D(\boldsymbol{A}, \boldsymbol{B}) := \min_{[y_{N,1}, \ldots, y_{N,k}, \ldots, y_{N,K}]} g((N, y_{N,1}), \ldots, (N, y_{N,k}), \ldots, (N, y_{N,K}) \mid N)$$

**Figure 3. DP algorithm for the PL2DW.**

$D(\boldsymbol{A}, \boldsymbol{B})$ gives a distance between $\boldsymbol{A}$ and optimally warped $\boldsymbol{B}$, i.e., $\{b(x(i,j), y(i,j))\}$.

In the calculation of (7), the nearest integer coordinates $(\lfloor x(i,j) + 0.5 \rfloor, \lfloor y(i,j) + 0.5 \rfloor)$ are used instead of non-integer coordinates $(x(i,j), y(i,j))$ given by (3).

### 2.4. DP algorithm

A DP algorithm for the PL2DW is described in Fig.3. This algorithm searches for the optimal pivot mapping $\{(x_{i,k}, y_{i,k})\}$ which specifies the optimal PL2DW between $\boldsymbol{A}$ and $\boldsymbol{B}$ as the optimal state transition sequence of the *multistage decision process* where stage, state, and state transition cost are denoted as $i$, $((x_{i,1}, y_{i,1}), \ldots, (x_{i,k}, y_{i,k}), \ldots, (x_{i,K}, y_{i,K}) \mid i)$, and $d((x_{i,1}, y_{i,1}), \ldots, (x_{i,k}, y_{i,k}), \ldots, (x_{i,K}, y_{i,K}) \mid i)$, respectively. Step 5 in Fig.3 is the so-called DP-recursion where $\Delta_k = \{(p_k, q_k) \mid p_k = 0, 1, 2,\ q_k = -1, 0, 1\}$ for $1 < k < K$ and $\Delta_k = \{(0,0), (1,0), (2,0)\}$ for $k = 1$ and $K$.

While back pointer and backtracking operations to obtain the optimal pivot mapping are omitted from the algorithm description of Fig.3, the readers can easily provide these operations.

### 2.5. Computational complexity

The computational complexity of the above DP algorithm is $O(NW^{2K}(9^K + N))$ where $W = 2w + 1$. Although this is an exponential order of $K$, the PL2DW generally requires far less computations than the conventional monotonic and continuous 2DW [8, 9] whose computational complexity is an exponential order of image size $N$. This is because for many patterns some small $K$ is sufficient to represent their shape variations, as is shown by the experiments in the next section. In such cases, the PL2DW becomes tractable enough even if image size $N$ is large.

It should be noted that the DP algorithm for the PL2DW problem is not unique. In fact, by referring [9], another (slightly complicated) DP algorithm where stage is denoted as $(i, k)$ instead of $i$ can be developed. If this DP algorithm is used, the time complexity is reduced to $O(NW^{2K}(9K + N))$. In the following experiment, this DP algorithm and its approximation algorithm were used.

## 3. Experimental Results

Figure 4 shows examples of the PL2DW obtained between $64 \times 64$ gray scale images with uniform backgrounds. The warp range $w$ was fixed at 10. Under this condition, Sun Ultra2 (SPECint_95:12.3, SPECfp_95:20.2) required about 2.5 and 150 seconds for $K = 2$ and $K = 3$, respectively. It should be noted that it is practically impossible to obtain the conventional monotonic and continuous 2DW [8, 9] as well as the monotonic 2DW [5], between a pair of $64 \times 64$ images even when $w = 1$. For (g), where $K = 5$, an approximation algorithm incorporated with beam search technique was used. In this algorithm, search width, i.e., the number of active states on each stage was limited to 1000 according to their cumulative cost $g$. Sun Ultra2 required about 4.5 seconds for this algorithm.

The results (a)–(d) suggest that the PL2DW often provides sufficient matching even when $K$ is very small. Especially in (d), only two pivots were arranged on the both sides of each row of $\boldsymbol{A}$. While this result shows slightly less accurate matching than (c), the slant of the characters on $\boldsymbol{B}$ was reasonably corrected.

The results (e)–(g) show the relation between pivot arrangement and warping accuracy. The difference between (e) and (f) suggests that the position of pivots affects the warping accuracy. In (f), since pivots were not arranged on the center of the face, the distortion caused by the lineariza-

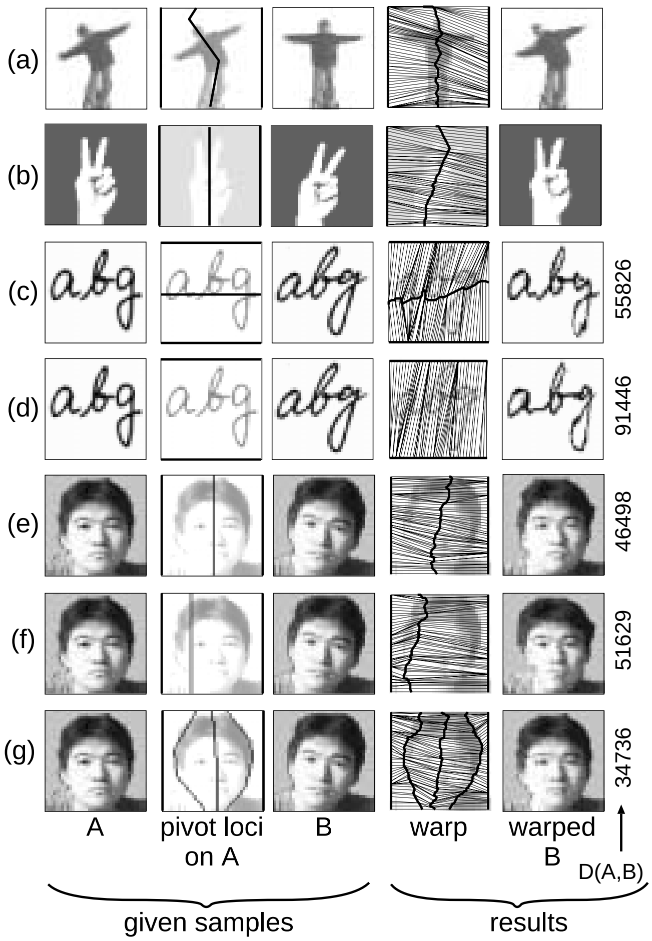


**Figure 4. Experimental results of the PL2DW.**

tion becomes evident. The difference between (e) and (g) suggests that the increase of the number of pivots improves the warping accuracy. In fact, the distance $D(\boldsymbol{A}, \boldsymbol{B})$ of (g) was smaller than that of (e).

In (g), the first and $K$th (i.e., 5th) pivots are not important. When these two side-pivots are removed, the PL2DW works as an elastic image pattern spotting technique.

## 4. Conclusion

In this paper, a DP-based algorithm for piecewise linear two-dimensional warping (PL2DW) was proposed. In the PL2DW, the mapping of each column of one image to another is given by linearly interpolating the mappings of several pivots prearranged on the column. Because of this linearization, the complexity of the present DP algorithm becomes far less than that of the previous DP-based 2DW algorithm [9]. While the linearization slightly limits the fitting ability of the 2DW, experimental results showed generally sufficient matching. This is because the pivots which guide the linearization can be prepared according to the knowledge related to image patterns to be matched.

**Acknowledgement:** This work was supported in part by the Ministry of Education, Science, Sports and Culture in Japan under a Grant-in-Aid for Scientific Research No.10680385.

## References

[1] A. K. Jain, Y. Zhong, and M.-P. Dubuisson-Jolly, "Deformable template models: A review," Signal Processing, vol. 71, no. 2, pp. 109–129, Dec. 1998.

[2] C. A. Glasbey and K. V. Mardia, "A review of image-warping methods," J. Applied Statistics, vol. 25, no. 2, pp. 155–171, Apr. 1998.

[3] H. Lester and S. R. Arridge, "A survey of hierarchical non-linear medical image registration," Pattern Recognition, vol. 32, no. 1, pp. 129–149, Jan. 1999.

[4] A. Redert, E. Hendriks and J. Biemond, "Correspondence estimation in image pairs," IEEE SP Mag., vol. 16, no. 3, pp. 29–46, May 1999.

[5] E. Levin and R. Pieraccini, "Dynamic planar warping for optical character recognition," Proc ICASSP, pp. III 149–152, 1992.

[6] I.J. Cox, S.L. Hingorani, S.B. Rao and B.M. Maggs, "A maximum likelihood stereo algorithm," Comput. Vis. Image Und., vol. 63, no. 3, pp. 542-567, May 1996.

[7] S. Kuo and O. E. Agazzi, "Keyword spotting in poorly printed documents using pseudo 2-D hidden Markov models," IEEE Trans. PAMI, vol. 16, no. 8, pp. 842–848, Aug. 1994.

[8] S. Uchida and H. Sakoe, "A monotonic and continuous two-dimensional warping based on dynamic programming," Proc. 14th ICPR, vol. 1 of 2, 521–524, 1998.

[9] S. Uchida and H. Sakoe, "An efficient two-dimensional warping algorithm," IEICE Trans. Info. & Syst., vol. E82-D, no. 3, pp. 693–700, March 1999.